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Note

A Stopping Method for Racah's Formulas*

There is a formula due to Racah for computing the inner multiplicity n_μ^λ of a weight μ of an irreducible representation with highest weight λ of a complex simple Lie algebra. It is [8, p. 23]

$$n_\mu^\lambda = - \sum_{\substack{w \in W \\ w \neq \text{identity}}} (\det w)\, n^\lambda_{\mu+(\delta - w\delta)} \tag{1}$$

where W is the Weyl group of the Lie algebra and $\delta = \sum_{i=1}^{l} \lambda_i$ is the sum of the fundamental irreducible representations.

There is also a formula due to Racah for computing the outer multiplicity $m^\lambda_{\lambda'\lambda''}$ of an irreducible representation with highest weight λ in the tensor product of the irreducible representations with highest weights λ' and λ''. It is [8, p. 22]

$$m^\lambda_{\lambda'\lambda''} = \sum_{w \in W} (\det w)\, n^{\lambda'}_{\lambda+\delta-w(\lambda''+\delta)}\,. \tag{2}$$

In most cases only a small number of the summands in either formula are nonzero. In fact in (1) we must have

$$\mu + (\delta - w\delta) \leqslant \lambda \tag{3}$$

for $n^\lambda_{\mu+(\delta-w\delta)}$ to be nonzero. Likewise in (2) we must have

$$\lambda + \delta - w(\lambda'' + \delta) \leqslant \lambda'. \tag{4}$$

In (3) and (4) the inequality is defined by the following requirement:

$$\mu \leqslant \eta \qquad \text{if} \quad \eta = \mu + \sum_{i=1}^{l} k_i \alpha_i\,,$$

where $\{\alpha_i\}$ is a system of simple roots and $k_i \geqslant 0$ for all i.

We now develop a recursive method of generating the set $\{\delta - w(\lambda'' + \delta) \mid w \in W\}$ and show how this method may be used to truncate the sums in (1) and (2) after a small number of terms.

Recall that the weights and roots belong to the real vector space $H_R{}^*$ and that $\{\lambda_i\}$ and $\{\alpha_i\}$ are bases for this vector space [2, pp. 68, 93]. To convert from the

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α-basis to the λ-basis one multiplies by the Cartan matrix $A = [a_{ij}]$ [2, p. 93]. Recall also that $s_j(\alpha_i) = \alpha_i - a_{ji}\alpha_j$ [5, p. 126] and that $s_j(\delta) = \delta - \alpha_j$ [5, p. 248], where s_j is the Weyl reflection defined by α_j. Let $\mu \in H_R{}^*$ and define the *α-level of μ* to be $\sum_{i=1}^{l} k_i$ when $\mu = \sum_{i=1}^{l} k_i\alpha_i$.

The recursive method of generating $\{\delta - w(\lambda'' + \delta) \mid w \in W\}$ is based on the following lemma.

LEMMA. *Assume the α-level of $\delta - w(\lambda'' + \delta) = \sum_{i=1}^{l} m_i\lambda_i$ is r. If $1 \leqslant j \leqslant l$, then the α-level of $\delta - s_jw(\lambda'' + \delta) = \sum_{i=1}^{l} (m_i + (1 - m_j))\, a_{ij}\lambda_i$ is $r + 1 - m_j$.*

Proof. Write $\delta - w(\lambda'' + \delta) = \sum_{i=1}^{l} k_i\alpha_i$, where $\sum_{i=1}^{l} k_i = r$. Rearranging and using the definition of s_j we have

$$s_jw(\lambda'' + \delta) = \delta - \alpha_j - \sum_{i=1}^{l} k_i(\alpha_i - a_{ji}\alpha_j).$$

Consequently,

$$\delta - s_jw(\lambda'' + \delta) = (1 - m_j)\, \alpha_j + \sum_{i=1}^{l} k_i\alpha_i. \tag{5}$$

Thus, in going from $\delta - w(\lambda'' + \delta)$ to $\delta - s_jw(\lambda'' + \delta)$, the α-level has been incremented by $(1 - m_j)$. Finally, we note that the expression for $\delta - s_jw(\lambda'' + \delta)$ in the λ-basis is obtained from (5) by multiplying by the Cartan matrix.

In using either of the Racah formulas for actual computation, a reasonable method of approach is to generate the set $\{\delta - w(\lambda'' + \delta) \mid w \in W\}$ and list it in a table. (For inner multiplicities, we set $\lambda'' = 0$.) One method of generating such a listing is to first generate the Weyl group and then apply each element of the Weyl group to $\lambda'' + \delta$. [1, 4–7] But for ranks $\geqslant 6$ the Weyl group becomes unmanageably large and as indicated above typically only a small number of table entries actually contribute to the sum in a given computation.

The following approach will work for large ranks as long as the dimensions of the representations are not too large. To generate the table, first note that for w = identity, $\delta - w(\lambda'' + \delta) = -\lambda''$, and make this the first table entry. Also record the fact that $\det(w) = 1$. We generate the table in order of increasing α-level. If $\zeta = \sum_{j=1}^{l} p_j\lambda_j$ already appears in the table, then for each j such that $p_j \leqslant 0$, we use the lemma to find another table entry. Its determinant entry will be the negative of the determinant entry for ζ. This process may give duplications but they can easily be located since they will only occur within a given α-level. The entire table will have been generated when $\zeta = \sum_{j=1}^{l} p_j\lambda_j$ with $p_j > 0$ for all j occurs in the table. (Any application of the lemma to this ζ will decrease its α-level.)

But usually we can stop before the entire table is generated. In fact, for the Racah inner formula we need only those table entries $\delta - w\delta$ such that $\delta - w\delta \leqslant \lambda - \mu$ for given λ and μ. If we assume that the multiplicity of a nondominant weight will be calculated by the use of equivalence classes, then we may restrict $\mu \geqslant 0$. Hence, for a given representation the table need only include $\delta - w\delta$ if $\delta - w\delta \leqslant \lambda$. A weaker but more easily computed condition is

$$\alpha\text{-level}\ (\delta - w\delta) \leqslant \alpha\text{-level}\ (\lambda). \tag{6}$$

A similar argument for the Racah outer multiplicity formula will give the stopping condition

$$\alpha\text{-level}\ (\delta - w(\lambda'' + \delta)) \leqslant \alpha\text{-level}\ (\lambda'). \tag{7}$$

We see then that using the recursive procedure described above together with the appropriate stopping condition ((6) or (7)) completely eliminates the necessity of first generating the Weyl group to use Racah's formulas. Since the table is generated in increasing α-level order, the conditions (6) and (7) are very easy to check for each new table entry.

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